On Leibniz’ Characteristic Numbers

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ABSTRACT: In 1679, Leibniz wrote nine manuscripts on three different arithmetical models of Aristotelian logic. This was a part of his project of a “calculus universalis”. First we show the precise relations of these three models to each other by presenting three criteria which serve the purpose of classifiying models of Aristotelian logic. This facilitates the understanding of Leibniz’ constructions. Our method is of special value for the sophisticated third model, the domain of which consists of pairs of natural numbers. We present a simple approach to Leibniz’ definitions which on first sight appear complicated. We show that it is possible to deduce, from the “universal positive” relation **a (** = “All ...”), the other three “Aristotelian relations” **i, o,** and **e**. – It has always been difficult to understand the exact nature of Leibniz’ characteristic numbers because of his misleading nomenclature, since he utilized the signs + and – in order to designate the “positive” and “negative” part of a characteristic number pair. We present a new interpretation of Leibniz’ symbolism, showing that the number pairs should be interpreted as numerator and denominator of a rational number. Thus we can identify the last model as the natural extension of the first and second one, showing the continuity in Leibniz’ different attempts towards an arithmetization of logic. – We close our paper by discussing two well known problematic aspects of Leibniz’ characteristic numbers, formulating two open questions concerning the formal structure of the system.

ZUSAMMENFASSUNG: Im Rahmen seines Projektes eines „calculus universalis“ entwarf Leibniz Anfang 1679 in neun Texten drei unterschiedliche Modelle der aristotelischen Logik mit Hilfe von Zahlen. Durch diese von ihm erfundenen „charakteristischen Zahlen“ wollte Leibniz die Schlussweisen der aristotelischen Logik auf rein arithmetische Rechnungen reduzieren. In der vorliegenden Arbeit zeigen wir genau, wie die drei Modelle untereinander zusammenhängen bzw. aufeinander aufbauen. Zu diesem Zweck geben wir drei Kriterien an, mit Hilfe derer sich Modelle der aristotelischen Logik klassifizieren lassen. Zum einen können wir dadurch die Leibnizschen Definitionen leicht nachvollziehen sowie auch die Stärken und Schwächen der einzelnen Modelle präzise beschreiben. Besonders für das ausgefeilte letzte Modell, dessen Grundbereich aus Paaren natürlicher Zahlen besteht, liefert unsere Methode einen ganz natürlichen Zugang zu den Leibnizschen Definitionen, die im Original recht sperrig wirken. Wir zeigen insbesondere – was Leibniz nicht herausgestellt hat - , dass allein aus der universell-positiven alle anderen drei „aristotelischen Relationen“ herleitbar sind. Ein Grund für manche Schwierigkeiten mit dem Verständnis der Leibnizschen charakteristischen Zahlen ist die von ihm verwendete Nomenklatur. Der problematischen Schreibweise mit Vorzeichen + und - , die in der Vergangenheit manchen Kommentator auf die falsche Fährte gelockt hat, geben wir eine neue Deutung: Dadurch entpuppt sich das letzte Modell als ganz natürliche Erweiterung des vorangehenden vom Grundbereich der natürlichen in den Bereich der positiven rationalen Zahlen. – Mit der vorliegenden Arbeit hoffen wir, der Verwendung charakteristischer Zahlen als theoretisches Instrument für die Untersuchung der aristotelischen Logik einen neuen Impuls geben zu können. Deshalb diskutieren wir am Schluss der Arbeit noch die aus der Literatur bekannten Stärken und Schwächen des Leibnizschen Modells und formulieren zwei offene Fragen zu diesem Komplex.

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1. Introduction

In a small number of unpublished manuscripts[1] from the spring of the year 1679, Leibniz invented a new method of arithmetization of Aristotelian logic. These texts belong to his gigantic project of a *calculus universalis*, which he described as follows:

> If one could find characters or signs, apt to express all our thoughts as purely and clearly as arithmetic expresses numbers or analytic geometry expresses lines, one could perform in all subject matters, as far as they are liable to rational thought, what can be done in arithmetic and geometry.[2]
>
> In order to establish a general calculus one has to find characters for arbitrary terms, by which, once they have been connected, the truth of the sentences composed by these expressions can be realized. I found that numbers are the most convenient characters. They are easy to work with and adapt themselves to all subjects, furthermore, they provide certainty.[3]

This quotation reveals that Leibniz is the progenitor of some of the main fantasies of our computer age! But this is not the subject here. Instead we will rather examine a particular *method* which Leibniz invented hoping that it would give him the key to replacing rational reasoning by algorithms based on numbers.

In the context of his arithmetization of Aristotelian logic, Leibniz employs an *intensional* approach[4] which means that his characteristic numbers stand for terms, denoting *concepts* (and not for sets of individuals, as in extensional logics). He considers the following types of propositions (*propositiones*) between terms (we are going to use Leibniz' notation[5]):

U.A.	propositio Universalis Affirmativa, („All *x* are *y*", *Axy*)
P.A.	propositio Particularis Affirmativa, („Some *x* are *y*", *Ixy*)
P.N.	propositio Particularis Negativa, („Some *x* are not *y*", *Oxy*)
U.N.	propositio Universalis Negativa, ("No *x* are *y*", *Exy*).

Leibniz is interested in the construction of a concrete model of Aristotelian logic which consists of numbers. These numbers, belonging to a fixed domain, are substitutes of the abstract terminal symbols *x, y*, ...On this domain, four concrete arithmetical relations ***a, i, o***, and ***e*** will assume the role of the logical symbols *A*, *I*, *O*, and *I* of Aristotelian logic, respectively[6].

The main question is: *Which* domain of numbers – together with which quadruple of relations – qualifies as a model for Aristotelian logic? After answering this question in Section 2, we shall be able to understand and classify Leibniz' different systems of characteristic numbers in Section 3

In his nine manuscripts on the subject of characteristic numbers, Leibniz does not deliver a recipe for the construction of suitable models but simply presents three which differ with respect to the underlying domain of numbers and the definition of the "Aristotelian relations".

[1] Leibniz: „Sämtliche Schriften und Briefe", Akademie-Ausgabe, 6. Reihe, 4. Band, Teil A, Berlin 1999, which we shall quote as **A**. The important manuscripts on the subject of the present paper are N. 56 to N. 64.We will also refer to the following translations: FS (N. 56 - N. 61, N. 63), P (N. 57, N. 63) and AG (N. 60, N. 62 and N. 64).
[2] **A**, N. 1, „La vraie methode", p. 6; FS, p.90.
[3] **A**, N. 59, p. 217; FS, p. 203.
[4] For a detailed discussion of the intensional and extensional aspects of Leibniz' logic see Kauppi 1960. Leibniz himself made some unequivocal remarks that he designed his characteristic numbers in the framework of an intensional interpretation of Aristotelian logic. In other parts of his works on logic he uses extensional concepts as well.
[5] The four different *propositiones* correspond to the Aristotelian predications which are classically denoted by *A, I, O, and E.*
[6] The relevancy of the notation *A* / ***a***, *I* / ***i***, *O* / ***o***, ***and*** *E* / ***e*** will be made clear in Section 2.

Thus, within his first two models, he utilizes the domain of natural numbers, and he chooses the usual relation of divisibility as interpretation of the *U.A.- propositio*:

> U.A.: Si Propositio Universalis Affirmativa est vera, necesse est ut numerus subjecti dividi possit exacte seu sine residuo, per numerum praedicati.[7]

The idea of this definition is that the relation of divisibility between numbers mirrors the relation between *genus* and *species*[8]: Genus (the *predicate* of a proposition) corresponds to the *part,* while species (the *subject*) corresponds to the whole. Accordingly, to the "higher" concept there belongs the smaller number which divides the larger number corresponding to the "lower" concept (the whole)[9]. Here we recognise clearly that Leibniz was aiming at a pure intensional calculus!

The remaining three *propositiones* are also reduced to the divisibility relation – we shall specify this in Section 3.

In order to illustrate his idea, Leibniz, in his first paper on this subject[10], presents the following example:

> Exempli causa, si fingeretur terminus animalis exprimi per numerum aliquem 2 (vel generalis a) terminus rationalis per numerum 3 (vel generalis r) terminus hominus exprimetur per numerum 2·3, id est 6, seu productum ex multiplicatis in vicem 2 et 3 (vel generalius per numerum a·r).[11]

The first two models which we shall denote by *A1* and *A2* respectively, have as underlying domain the set of natural numbers[12]; they differ in their respective definitions of the *P.A.* – and *U.N.- propositiones.* Leibniz' third and last model (often called *the* model of characteristic numbers) will be denoted by *Model B* in this paper. Its underlying domain is a certain set of *pairs* of numbers[13].

In this way we will obtain a first rough classification of the three number models by means of the type of the underlying number domain. However, there is a finer classification by fundamental properties of the four "Aristotelian relations". We shall present three criteria to be fulfilled by the relations ***a, i, o,*** and ***e*** in order to qualify as Aristotelian relations. These criteria mirror the following three well known properties of classical Aristotelian logic:

Criterion 1: validity of the central classical syllogism, *Barbara.*

Criterion 2 : formalization of Aritotle's method of reasoning by *ecthesis.*

Criterion 3: contradiction, a main part of the square of oppositions.

[7] **A,** N. 56, S. 182. „It is necessary for the universal positive proposition to be true, that the subject number can be divided exactly without remainder by the predicate number".
[8] **A**, N. 57, p. 199 –200.
[9] „... ita ut generis notio sit pars, speciei notio sit totum, componitur enim ex genere et differentia." (**A**, N. 57, 11, p. 199).
[10] N 56, 17, p. 182.
[11] „If, for example, we assume that the item ‚animal' is expressed by means of the number 2 (or, in general, by a), and the item ‚rational' by means of the number 3 (or, in general, by r), then ‚man' is expressed by 2·3, i.e. 6, as the result of the product of 2 by 3 (or, in general, by the number ar)".
[12] Here we do not include zero in the set of natural numbers.
[13] In Section 3 we shall see that we could as well say: The underlying domain of model B consists of the set of *all rational numbers.* As neither Leibniz nor any of his later commentators was aware of or mentioned this observation, we continue to talk about number pairs. Later we shall elucidate the connection of Leibniz' Model B with the set of rational numbers.

We will also discuss another method of justification of these three criteria, referring to modern model theory of Aristotelian logic[14]. This is not surprising, because Leibniz' manuscripts deal with quite concrete models of Aristotelian logic, and it is just model theory which allows us to examine the relation between models (semantics) and syntax of a logical theory. In Section 2, we will show that the three criteria mentioned above are quite natural assumptions in the framwork of model theory[15].

Both approaches to Leibniz' system lead to the same classification scheme[16] which we resume in Table 1:

Name of model	Reference to Leibniz' papers	Basic number domain	Criteria satisfied?
A1	N. 56	natural numbers	no
A2	N. 57, 58, 59	natural numbers	yes
B	N. 60 - 64	pairs of natural numbers	yes

Table 1

This table does not show *which* of the three criteria is not satisfied in Model A1, neither does it reveal *why* Leibniz enhanced Model A2 in spite of its compliance to all criteria. In Section 3 we will see that though A2 is formally qualified as a model of Aristotelian logic it is not comprehensive enough to be of any practical use.

By means of this paper we aim to disclose the exact formal structure of Leibniz' three number models and also to elucidate the precise relation of these models to each other[17]. We shall see that A1 and A2 differ fundamentally in that only A2 complies with the "canonical" property of *ecthesis* which links the universal positive (***a***) to the particular positive proposition ***i***.

Aristotle employs εκθεσις (ecthesis) as a method of proof for some syllogisms (*Baroco*, *Bocardo*[18]) as well as for the proof of the ***e*** - *conversion*[19]. One can also use ecthesis as an alternative to a proof by *reductio ad absurdum* in certain formal reconstructions of Aristotelian logic[20].

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[14] This modern theory of Aristotelian logic by means of systems of natural deduction – not being based on predicate calculus - was founded by Corcoran, 1973 and, independently, by Smiley, 1973.

[15] There exist alternatives to our way of proceeding, in particular with respect to our emphasis on *ecthesis* (e.g. Lukasiewicz ,1951). But it is not our aim to discuss all possible kinds of formalization of Aristotelian logic, but just to develop a special set of instruments for the purpose of understanding and classifying Leibniz' research on characteristic numbers.

[16] Readers who are not interested in modern model theory may skip the somewhat formal Section 2 without missing a central point of our classification and interpretation of Leibniz' number models.

[17] At first sight N. 61 seems to contain an additional „transitional model" of type A2/B, where Leibniz took the *U.A.*- and *P.A. propositiones* from Model A2, and the remaining *propositiones* from Model B. However, this is not possible by pure formal reasons, as the basic number domains of A2 and B are different (cf. Table 1).The whole matter is clarified by commentaries in the Akademie edition, p. 228 and p. 233.

[18] An. Pr. 30a6- 14.

[19] „Now, if A belongs to none of the Bs, then neither will B belong to any of the As. For if it does belong to some (for instance to C), it will not be true that A belongs to none of the Bs, since C is one of the Bs." (An. Pr. 25a15- 19; transl. by R. Smith, 1989). Here C denotes the term constructed by exposition (*ecthesis*) which appears neither in the premises nor in the conclusion of the proposition. – Since its invention by Aristotle, this method led to considerable confusion which arose mainly regarding the question of „ontological status" of the „exposed" term C: Does C denote an individual or a concept? Meanwhile, there are diverse proper formalizations of this method (cf. Smith, 1982). – Burkhardt, 1980 writes: „Die Ekthese als Beweis eines Syllogismus findet sich bei Leibniz nicht (Ecthesis as means of proof for syllogisms cannot be found in Leibniz' works)". But we shall see that Leibniz explicitly refers to ecthesis in his construction of the *P.A. propositio,* even he does not mention the name of the method.

[20] Robin Smith, 1982.

In Model A2, the ***a*** – relation and the ***i*** – relation are connected via ecthesis (contrary to Modell A1). Therefore A2 fulfills all conditions of a model in the sense of modern model theory of Aristotelian logic[21]; we will present the details in Section 2.

Concerning model B we will show that Criteria 1 to 3 imply that only one of the four *propositiones*, the *U.A.- propositio*, can be chosen independently whilst the others are subordinated. This is of some advantage, as the U.A. – proposition, representing exactly Leibniz' idea of employing divisibility, is the simplest one[22].

We shall also attempt to make clear that Leibniz' notation +s-σ for his number pairs in model B is not a good choice as it induces misleading associations with pairs of positive and negative numbers. In opposition to that we will show in Section 4 that the correct way of interpretating the characteristic numbers of model B is to regard them as positive rational numbers s/σ.

It is well known that Leibniz ceased to work on the subject of characteristic numbers after having written the manuscript **A,** N. 64. As far as I know, he did not give reasons for abandoning this project. Whereas it has sometimes been said that the system is faulty[23], we know on the contrary, since the work of Lukasiewicz[24], that model B is, in a certain sense, even *perfect*: It is a model in which *exactly*[25] the syllogistic deductions of classical Aristotelian logic hold true.

Thus there exist no internal formal errors which are responsible for Leibniz' abandoning of his efforts on the project of characteristic numbers[26]. But there is evidence that Leibniz, trying to incorporate *negative concepts* into his formalism reached a point of research where, as we shall see, there was no chance for him to succeed. We also show that, in his last lines concerning this subject, he may even have got muddled by his own plus-minus-notation, struggling hard with a task which he could not resolve within his system. This may indeed have been the reason for his abrupt stop in working on the characteristic numbers – we don't know. In the last part of our paper we formulate an open question regarding this point of including "negative concepts" into the system of characteristic numbers.

Finally, we will point to an important open problem in the context of the theory of characteristic numbers: Leibniz never attacked the question of how to assign numbers to terms. While he presented some very small examples, he did not even mention the fact that one would need an algorithm for the computation of characteristic numbers in order to realize his dream of replacing thinking by computing. We will formulate this basic problem hoping to stimulate further research on this subject of the "Gödelization of Aristotelian logic"!

In the following Section 2 we give a short introduction into the basics of model theory applied to Aristotelian logic. Its purpose is to show why just the three criteria presented above are the important ones for any model of Aristotelian logic. The reader who is not interested in abstract model theory may skip the whole section without risk of not understanding the other parts of the present paper.

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[21] Leibniz described A1 as his first model but abandoned it in favour of A2 from N.57 on without further comment

[22] It is easy to see that within Model B the definition of the *U.A. – propositio* generates the definition of the P.N. – *propositio* by negation. In the same manner the definitions of the *P.A. – propositio* and the *U.N. – propositio* are related to each other. This choice of Leibniz conforms to the classical approach (c.f. our Criterion 3).However, our observation that one can even derive the *P.A.- propositio* (***i***) from the definition of the *U.A.- propositio* (***a***) is new.

[23] This view is due to Couturat, 1903 and since then has cut the surface from time to time. Thiel, 1980, correctly pointed out that this view is unfounded; cf. Henrich, 2002, concerning the history of reception of Leibniz' characteristic numbers.

[24] Lukasiewicz, 1951.

[25] The complicated part is to prove that, within B, there exist no additional valid syllogisms compared to Aristotelian logic; this is a deep result of Slupecki, a pupil of Lukasiewicz.

[26] Therefore also the work of Sotirov, 1999, leads in the wrong direction: He „corrects" model A2 with the result that, in his model, only finitely many numbers are of importance. This was not Leibniz' aim who constructed only systems with infinitely many numbers.

2. Models of Aristotelian logic

We will introduce only those basic facts about modern model theory, invented by Tarski, which are needed to speak precisely about syntactic and semantic aspects of Aristotelian logic[27]. First, one has to distinguish very precisely syntax and semantics as two completely different layers. In a second step one connects these concepts by suitable maps in order to obtain a completeness theorem[28], thus showing the equivalence of syntactic and semantic reasoning[29].

The whole theory of characteristic numbers belongs to the sphere of semantics of Aristotelian logic. This distinguishes Leibniz' theory from those earlier theories which are merely concerned with syntactical matters, as well as from the work of others which, like Aristotle[30], preferred to alternate continuously between syntactical and semantical argumentation[31].

With respect to Aristotelian logic, Corcoran[32] and, independently, Smiley[33] invented a calculus of natural deduction in 1973 and proved a completeness theorem. In continuation of these works, Martin[34] published a very general model theory of Aristotelian logic, presenting a completeness theorem which corresponds to Gödel's work on completeness of first order predicate logic.

Fortunately it is not necessary to step into the subtleties of a *special* calculus in order to understand Leibniz' work[35]. For our present purpose it is of more importance to understand the principal structure of such calculi in order to draw the proper consequences with respect to the semantic domain of Leibniz constructions.

Any calculus of this type has a fixed set of term constants (non logical constants) *x, y, z, ...* as basic objects which, in Aristotelian logic, denote concepts. In addition there are four *copula* (logical constants) *A, I, O*, and *E* which, together with two terminal symbols, are the compounds of Aristotelian propositions *Axy*, *Ixy* etc. The well formed formulas (wff's) of Aristotelian logic are expressions of type *Uvw* where *v* and *w* stand for term constants, and *U* is one of the four copula.

Now a calculus - such as the aforementioned one by Martin[36] - contains a certain number of *rules* which, starting from a given set ***?*** of wff's (propositions), allow to derive new wffs's. For example, any such calculus will contain a rule allowing the derivation of the new wff *Axz*, given that *Axy* and *Ayz* already belong to Σ (*Barbara* –syllogism of Aristotelian logic[37]).

[27] We treat only the basic ideas and the terminology of model theory, not any theoretical results. This model theoretic view on Leibniz' characteristic numbers was used by Thiel, 1980 in order to correct wrong interpretations of Leibniz logic. Thiel did not explicitly mention model theory but argued from its standpoint.
[28] It is well known that in 1929 Gödel proved a completeness theorem for the first order predicate logic.
[29] Exactly this split between syntax and semantics is Tarski's great achievement. Although there exist alternatives to Tarski's approach, we emphasize that his theoretical framework is particularly well suited for the context of Leibniz' arithmetical models.
[30] Aristotle seems to have been perfect in mastering the syntax - semantics – game: „Aristotle is well aware of a distinction between syntax and semantics, indeed, in a way familiar to Church, Tarski and other modern logicians“ (Boger 1998, p.195).
[31] This kind of change of frames is quite normal and formal correct in case that one has a completeness theorem for the calculus under consideration.
[32] Corcoran, 1972.
[33] Smiley, 1973.
[34] Martin, 1997.
[35] There exists different systems of syntax which try to capture Aristotle's formal logic of the *Analytica priora,* cf. Thom, 1981.
[36] Martin, 1997.
[37] Lukasiewicz was the first to present a sound formalization of Aristotelian logic without reference and in sharp contrast to the „modern“ extensional view on Aristotle of the 19th century. His method differed from the one employed by Corcoran in that he did not regard syllogisms as rules of deductions but as conditional sentences used as axioms of his theory. - Today we know that the interpretation of Aristotelian logic as a logic of relations of classes of individuals is not correct, even if it appears in almost all modern textbooks. In his famous book on Aristotelian logic, Lukasiewicz tried to convince his readers that this extensional interpretation does not comply with Aristotle's aims, but not even a renowned logician like him succeeded in eliminating the misinterpretation of the early ages of modern logic. It seems that Leibniz was closer to Aristotelian logic than the protagonists of the modern class- and predicate – calculus of the

By $\boldsymbol{A}(\Sigma)$ we denote the “Aristotelian closure” of *?*: $\boldsymbol{A}(\Sigma)$ contains all those propositions which one can deduce syntactically from *?* by iteratively applying the rules of the given system[38].

Concerning semantics, we start with a nonempty set ***S***, the basic domain (in Leibniz’ two models of type A, ***S*** is the set of natural numbers). There are four two-place relations defined on ***S*** which we will denote by ***a, i, e***, and ***o.*** Now it is very important to distinguish these concrete relations from the symbols *A, I, E* and *O* which appear at the syntactical level! This distinction is a prerequisite for the possibility to argument in terms of models. While the copula are fixed constituents of the syntactic calculus, ***a, i, e***, and ***o*** denote relations on a given set ***S***[39]. Of course, these relations will have to possess certain properties relative to Aristotelian logic, a subject which we will discuss subsequently.

First we shall see how the relationship between the syntactic and the semantic domain will be established. The important concept is that of an *interpretation*: Let there be given a function *R*, assigning to each terminal constant *x* a certain element *?=R(x)* of the basic domain ***S***. Each proposition will get automatically, via this function R, a truth value by means of the following construction: A proposition (wff) of type *Axy* gets the truth value *true* or *false*, depending on whether the relation ***a*** holds true for the elements *R(x), R(y)*[40]. Analogously one defines the interpretation of the other propositions *Ixy, Oxy,* and *Exy,* employing the relations ***i, o*** and ***e,*** respectively.

Let there be given a set ***?*** of propositions and a basic domain ***S***, together with four concrete relations ***a, i, o*** und ***e*** on ***S***. Then (***S***,*R*,***a***,***i***,***o***,***e***) will be called a *model* of ***?,*** if all propositions of ***?*** become true by means of the interpretation specified above.[41] [42]

Now we turn to the core problem: Which kind of properties of the relations ***a, i, o*** and ***e*** have to be true in order that ***S***, together with these relations, qualifies as a possible domain for Aristotelian logic? That these relation cannot be defined in complete independence of each other may be seen by the following reasoning:

The explicit aim of model theory is to construct a certain consonance between syntax and semantics. Now, on the syntactic level, where one constructs the Aristotelian closure ***A***(***?***) of a set ***?*** of propositions, everything is fixed by the rules of the calculus. These rules, however, do not enter into the definition of the basic domain ***S*** and of the four relations on ***S***. Thus, in order to make such a consonance of syntax and semantics possible (which, if it holds true, will be expressed by the completeness theorem), one has to impose certain conditions on the relational structure which mirror the main properties of the syntactic calculus.

In an important paper from 1997, Martin presented a general framework into which he was able to embed Corcoran’s calculus of natural deduction. Generalising also Corcoran’s semantics, he further showed very convincingly that the “natural” structure for a semantic domain

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19th century. He always emphasized the difference between intensional and extensional logic and used the former explicitly in the framework of his arithmetical calculus. Couturat as well as Russell, criticising Leibniz for his utilization of intensional logic, did not realise that Leibniz had made his choice on this subject deliberately and out of good reasons.

38 For our present purpose these are the sole elements which are needed to understand of the syntax of Aristotelian logic: the structure of wff’s representing propositions and the concept of Aristotelian closure ***A***(***?***) of a given set ***?*** of wff’s (propositions). We do not go into the details of how a certain calculus enables us to construct the Aristotelian closure ***A***(***?***), i.e. the set of all consequences of given propositions ***?*** by means of the rules of Aristotelian logic. Completely different realisations have been given by Lukasiewicz (1951), Corcoran (1973), Brillowski (1992).

39 In order to be quite precise, we should index the relations by the symbol ***S*** but we don’t want to overdo it. From the mathematical standpoint, two place relations on ***S*** are subsets of ***S***×***S***.

40 The interpretation *R(A(x,y))* of the proposition *Axy* is the truth value of ***a****(R(x),R(y))*; we have *R(A(x,y))=true if and only if (R(x),R(y))*∈ ***a.***

41 A completeness theorem for a certain calculus is equivalent to the statement that ***?*** and ***A***(***?***) possess exactly the same models.

42 An example: Let ***?***={*Axy*,*Ayz*} and ***S*** = ***N*** (set of natural numbers), where ***a*** denotes the usual order relation on ***N***. We obtain a model of ***?*** by assigning to *x, y,* and *z* the numbers 1, 2, and 3, respectively.

of Aristotelian logic is a special type of partial ordered set, a so called “meet semi-lattice”[43]. In the following, we shall refer his construction, as it leads to a clear understanding of the semantics of Aristotelian logic, allowing to connect modern formal considerations with classical results.

As we just mentioned, Martin considers „order theoretic interpretations“ (in contrast to the „set theoretic interpretation“ of Corcoran), where ***S*** is a partial ordered set and where the given partial order relation on ***S*** is the natural candidate for the ***a***- relation. As any partial order satisfies the axiom of transitivity we are now able to state the first criterion to be satisfied for the ***a*** – relation on ***S***:

1. *Transitivity:* ***a*** has to be a transitive relation; i.e., given *?, ?, ?* in ***S*** such that ***a***(*?, ?*) as well as ***a***(*?, ?*) hold true, then it follows that ***a***(*?, ?*) is also valid[44].

The transitivity of the relation ***a*** guarantees that the central syllogism of Aristotelian logic, *Barbara*, has its correlating counterpart in semantics[45].

The second criterion refers to the “particularly positive” relation ***i:***

2. *Ecthesis*[46]: ***a*** and ***i*** are linked by the following property: For any two elements *?, ?* of ***S*** the relation ***i****(?,?)* holds if and only if there exists a further element *?* such that ***a****(?,?)* as well as ***a****(?,?)* holds true[47].

The third condition refers to the connection of the relations ***o*** and ***e*** to ***a*** and ***i,*** respectively: In the calculus of Corcoran / Martin, ***o*** is defined as the negation of ***a,*** and ***e*** is the negation of ***i***:

3. *Contradiction:* For every $\lambda, \mu \in$ ***S*** :

o(λ,μ) if and only if ***not a***(λ,μ); ***e***(λ,μ) if and only if ***not i***(λ,μ).

We emphasise that the partial order ***a*** is the central relation which allows us to define the other three Aristotelian relations. ***i*** is uniquely defined by condition 2, and then ***o*** and ***e*** are determined by means of condition 3 by ***a*** and ***i***, respectively[48].

In the following section we will check which of the three criteria are fulfilled for Leibniz’ models. One of our results will be that Criteria 1 and 3 will be satisfied in all cases. Therefore we shall focus on Criterion 2, *ecthesis*, which will help us to create a precise classification of all models.

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[43] A meet semilattice is a partial ordered set with a *meet operator* $\wedge$, where $x \wedge y$ denotes the infimum of *x* and *y*. Martin requires ***S*** to be a partial ordered set with a smallest element, denoted by 0. In order to adjust our setting to Martin’s formal requirements, we have to define an „artificial“ smallest element.

[44] For our classification of Leibniz’ models we need only the transitivity of the partial order relation, not the other defining properties, reflexivity and antisymmetry. Corcoran as well as Martin do not use reflexivity, because there is no evidence that Aristotle employed tautologies of type *a(? ?,)* . – One may enforce antisymmetry (***a****(?,?)* and ***a****(?, ?)* together imply *?=?*) by constructing suitable equivalence classes of elements of S. This will guarantee that there are no “loop” with the genus – species – relation.

[45] At first sight it seems astonishing that *Barbara* is the only syllogism to be introduced explicitly into the axiomatics of the semantic domain – all the more as one can prove that, in such a partial ordered domain, *all* syllogistic conclusions are valid! However, in the present paper we refrain from going into the details of syllogistic theory in order not to inflate it.

[46] Cf. Footnote 20, concerning the role of ecthesis in Aristotelian logic. In Section 3 we shall elucidate the role of *ecthesis* in Leibniz’ arithmetical logic.

[47] This connection between ***a*** and ***i*** corresponds to Martin’s definition as follows: The proposition *Ixy* obtains the value “true” in a model domain, if and only if $R(x) \wedge R(y) \neq 0$. Here $\lambda \wedge \mu$ denotes the greatest lower bound of λ and μ, i.e. the element in which λ und μ «meet below» (in the sense of the partial order); $\lambda \wedge \mu \neq 0$ signifies that there exists a $\xi \neq 0$ such that $\xi \leq \lambda$ and $\xi \leq \mu$, *which corresponds exactly to our Criterion 2.*- Lorenzen, 1995, expressed this connection between ***a*** and ***i*** very elegantly in the form of the equation ***i***=***aâ***; here ***â*** is converse to ***a:*** ***â***(η,φ):=***a***(φ,η).

[48] This corresponds to Kant’s view that the whole logic of Aristotle is based on the principle `nota notae est nota rei ipsius; repugnans rei ipsi` (Kant, Logic: §63).

3. Two models of type A

Leibniz' first two models A1 and A2 (cf. Table 1) belong to *one* group, because they are both based on the domain of natural numbers. This is in contrast to the third and final model B where Leibniz employed the idea of using pairs of numbers. In this respect, both models of type A are relatively simple, and are therefore suitable for demonstrating Leibniz' principal aims. But the major reason for going into the details of type A – models is our observation, the details of which we will discuss in Section 4, that the "perfect" but more complicated model B is an *extension* of A2. This observation saves us some troublesome calculations which would be necessary without a general theory putting the final model into a continuity with the preceding simpler models[49].

The first model, A1, appears at the beginning of Leibniz' first paper on characteristic numbers[50]. Then, from the following paper on, Leibniz replaces his definition of the ***i*-** proposition in A1 without indicating any motive for this change. This leads to the second model, A2. – We will now examine A1 as well as A2.

Leibniz basic idea was to assign to each "simple" concept a prime number and to each "composed" concept the product of prime numbers[51]. This idea leads directly to his definition of the „*U.A. propositio*" to which we referred in the Introduction. Written in formal terms:

U.A. propositio (Definition of relation ***a***): Let s and p denote positive integers. Then $(s,p) \in \boldsymbol{a}$ if and only if $p \mid s$[52].

Instead of "$(s,p) \in \boldsymbol{a}$ " we will write $\boldsymbol{a}(s,p)$, keeping in mind that the expression $\boldsymbol{a}(s,p)$ has the value "true" or "false" depending on whether p divides s or not. - The divisibility relation is well known to be transitive: if *m* divides *p* and *p* divides *s*, then *m* divides s. This implies that – for A1 as well as for A2 – the first Criterion of Section 2 (transitivity) applies to both models. As Leibniz' always took care that Criterion 3 was met by all his models[53], the decisive test will be Criterion 2.

In the following we present the two different definitions which Leibniz gave for the "*P.A. propositio*" (***i*-** relation) in A1 and A2, respectively.

Model A1. Definition[54] of relation $\boldsymbol{i}_1$: $\boldsymbol{i}_1(\lambda,\mu)$ holds, if and only if λ divides μ or μ divides λ.

Model A2. Definition[55] of relation $\boldsymbol{i}_2$: $\boldsymbol{i}_2(\lambda,\mu)$ holds if and only if there are numbers *n, m* such that $n\cdot\lambda = m\cdot\mu$.

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[49] The somewhat technical features of Leibniz' definitions which, without theoretical framework, look a bit complicated, may be responsible for the fact that, after Lukasiewicz and his pupil Slupecki, no one dealt with the formal aspects of the system of characteristic numbers on a general level.

[50] N. 56 p. 182 / 183 and p. 187.

[51] There exists clues (cf. Kauppi, 1960), that Leibniz' intended to restrict his number domain to the subset of so called *squarefree* numbers in which every prime factor appears only once (f.i., 18=2·3·3 is not squarefree because the prime factor 3 appears twice). As this distinction is of no relevancy for our present intentions, we will not employ this restriction of squarefreeness.- This remark applies also to the final model B discussed in the next section.

[52] p | s denotes the fact that p is an integer divisor of s (i.e., p divides s without remainder).

[53] This is quite natural if one takes into account the classical „square of oppositions" which includes the contradictions ***a/o*** and ***i/e.***

[54] "Si propositio Particularis Affirmativa est vera, sufficit ut vel numerus praedicati exacte dividi possit per numerum subjecti, vel numerus subjecti per numerum praedicati." (N. 56, p. 183) and "Sit propositio particularis affirmativa *quodd. A est H* (vel *qu. H est A*), ergo vel *H/A* aequ. *r* vel *A/H* aequ. *t* fiet *H* aequ. *rA* vel *A* aequ. *tH*." N. 56, p. 184 (12).

[55] "P.A. Qu. A est H, ergo rA aequ. vH". N. 59, p. 220.

These two definitions do indeed lead to different results: $\boldsymbol{i_1}(3,5)$ does not hold true while $\boldsymbol{i_2}(3,5)$ does (just choose $n=5$, $m=3$). $\boldsymbol{i_2}$ is an extension of $\boldsymbol{i_1}$ as from the validity of $\boldsymbol{i_1}(\lambda,\mu)$ we can deduce the validity of $\boldsymbol{i_2}(\lambda,\mu)$ (choosing n=1 or m=1), but not vice versa.

The important observation is that $\boldsymbol{i_2}$ conforms to Criterion 2: If there exist numbers n, m such that $n\cdot\lambda = m\cdot\mu$, then it suffices to define $\xi = n\cdot\lambda$ (= $m\cdot\mu$) which gives us the number sought in the formulation of Criterion 2 (*ecthesis*). Reversely, let there exist an "ecthesis number" ξ. By definition of ***a*** we have $\xi \mid \lambda$ as well as $\xi \mid \mu$. Hence, there exist numbers n , m such that $\lambda = n\cdot\xi$ and $\mu = m\cdot\xi$. By multiplying the first of these equations by m and the second by n, the equation $n\,\lambda = m\cdot\mu$, requested in the definition of $\boldsymbol{i_2}$, follows.

As $\boldsymbol{i_2}$ satisfies Criterion 2 this *cannot* be the case for $\boldsymbol{i_1}$, because ***i*** is *uniquely* determined by Criterion 2 by the basic relation ***a***.

Exactly this connection between the *U.A.*- and the *P.A. propositio* is the reason why Leibniz replaced the Model A1 with A2:

> Sed in propositione affirmativa particulari non es necesse ut praedicatum in subjecto per se et absolute spectato insit, seu ut notio subjecti per se praedicati notionem contineat, sed sufficit praedicatum in aliqua specie subjecti contineri seu notionem alicujus exempli seu speciei subjecti continere notionem praedicati; licet qualisnam ea species sit, non exprimatur.[56]

Later, in connection with the „Propositio particularis affirmativa" he writes:

> Sin species subjecti praedicatum continet ut partem, praedicatum erit genus speciei subjecti per art. 11. Itaque praedicatum et subjectum erunt duo genera ejusdem speciei.[57]

and at the end of his three manuscripts on model A2, in N. 59, p. 200 he presents the precise definition of the *P.A. Propositio* which we gave above.

Let us sum up: Leibniz tried to realize his idea of using natural numbers for modelling Aristotelian logic by means of two different models, A1 and A2 respectively. Both models comply with Criterion 1 (transitivity of ***a***) and Criterion 3 (definition of ***o*** and ***e*** in contradiction to ***a*** and ***i***). The models differ in that A1 does *not* fulfill Criterion 2 (ecthesis), and Leibniz constructed A2 deliberately in such a manner that relation ***i*** (*propositio particularis affirmativa*) satisfies the condition of ecthesis, Criterion 2.

Now that we have told the good news about Model A2, which, from a theoretical point of view, has all the features that are required to model Aristotelian logic, we have to explain why A2 is nevertheless useless:

By construction of the *propositio particularis affirmativa* $\boldsymbol{i_2}$, it follows immediately that for *any two arbitrary chosen numbers* λ and μ, $\boldsymbol{i_2}(\lambda,\mu)$ holds true - choose simply $n = \mu$ and $m = \lambda$! This is a clear disadvantage of the model, all the more because there is a consequence of this fact for the *propositio universalis negativa* (i.e., the ***e***- relation) which is due to the definition of ***e*** as negation of ***i*** (Criterion 3): There are no numbers ϕ, γ at all for which $\boldsymbol{e}(\phi,\gamma)$ holds true[58]. It is clear that such a model is useless for the purpose of modelling Aristotelian logic, and we un-

[56] N. 57, 18, p. 203: "It is not necessary that in the particular positive proposition the predicate is per se and absolute in the subject, but it suffices that the predicate is contained in any species of the subject or that the notion of any example or a species of the subject contains the predicate."Cf. also FS, p. 188/189.

[57] N. 57, 20, p. 204: But if a species of the subject contains the predicate as a part, then the predicate will be the genus of the species of the subject ... . Therefore, the predicate and the subject will be two genera of the same species.Cf. also FS, p. 190.

[58] This has already been mentioned by Couturat, 1903; cf. Kauppi , 1960.

derstand that Leibniz searched for more sophisticated, „richer" models. This will be the subject of the next section.

4. Characteristic number pairs

What are the possibilities of constructing a "better" model? If one adheres to the basic domain of natural numbers and to the definition of the *U.A. propositio* by divisibility, as well as to the fulfilment of the three aforementioned criteria, then there exists *no* model exept for A2[59]. The only remaining possibility is to enlarge the basic domain of the characteristic numbers. Thus one may hope to obtain a "richer" model, retaining all the positive properties of A2. This is the idea Leibniz pursued successfully.

Today we are well aware that there are two different standard possibilities of extending the realm of natural numbers. One may embed these numbers

- into the set of integers (positive and negative numbers) or
- into the set of positive rational numbers.

The basic method is identical in both cases: One constructs a new domain by introducing *pairs* of natural numbers, whereas the difference lies in the kind of operation one is focussed on: addition or multiplication. If the aim is to construct a domain where addition will be fully invertable, the solution is to enlarge the basic domain by negative numbers, thus arriving at the set of integers. If one aims at the possibility of unrestricted inversion of multiplication, then the construction will lead to rational numbers[60].

At first Leibniz speculated on amending the simple Model A2 by employing negative numbers, but he quickly rejected this idea. What he finally worked out was the second possibility, but he did not mention that his construction of characteristic numbers is connected to rational numbers, inasmuch as he used a notation which obscures this idea[61] for his readers if not for himself.

4.1 Negative numbers

Leibniz' experiment with negative numbers[62] is connected to an interesting logical aspect, namely, to Leibniz' hope to be able to model „negative concepts using negative numbers. If the positive number *m* corresponds to the concept „man", then one would like to have *–m* to denote „non-man"[63]. If such a construction were possible without contradiction, this would at once lead to an alternative for the definition of the critical ***e*** – proposition of model A2: In this case one could, as in classical logic, define

$$\boldsymbol{e}(\lambda,\mu) := \boldsymbol{a}(\lambda, -\mu),$$

a possibility which Leibniz was aware of[64]. But he rejected this on account of the following consideration: Let a term, having the characteristic number *m*, be composed of two other terms

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[59] Because Criteria 2 and 3, ***a*** defines ***i, o*** and ***e*** *uniquely*.

[60] From a mathematical standpoint both cases are examples of the construction of a group out of a semigroup without divisors of zero. In the first case the semigroup is the set of natural numbers (zero included) with addition, in the second case it is the set of non-zero natural numbers with multiplication.

[61] With the wisdom of hindsight it is easy to conjecture that the second possibility is the most promising one, because in A2 the basic relation ***a*** is defined by properties of multiplication/division, not of addition/negation.

[62] N. 59, p. 220, 10.

[63] Aristoteles did not use such negative terms, but later this construction belonged to the basic canon of classical logic.

[64] N. 58, p. 215, 16: "*Nullum cuprum est aurum*, id est *non quoddam cuprum est aurum*, ostendamus ergo tantum hanc propositionem falsam esse *quoddam cuprum est aurum*. Item *nullum est aurum*. Ergo *omne cuprum est non aurum*."

with negative numbers $-\lambda$ and $-\mu$, i.e. $m=(-\lambda)\cdot(-\mu)$. This would immediately imply $m=\lambda\cdot\mu$, meaning that m is also composed of λ and μ - which does not make sense! Another point[65] (not mentioned by Leibniz) is the fact that the basic law of contraposition of classical logic would not be valid: The obviously true proposition ***a***(6,2) would, by contraposition, imply ***a*** (-2,-6) which is false, because –6 does not divide –2!

Thus Leibniz was right in not using negative numbers for his models. It may, however, be that his experiments led him directly to his final Model B[66] which we are going to discuss in the following.

4.2 Rational numbers

Leibniz introduced the notation

$$+s - \sigma$$

for pairs of positive integers s, σ. This notation is problematic because, in Leibniz' theory of characteristic numbers, the tokens + as well as – *never* have the function of indicating a signed (positive or negative) number in the sense of mathematics! These „Notae[67]" fulfill the sole purpose of fixing the order of the two parts of an ordered pair of numbers. Today we would simply write

$$(s,\sigma).$$

Certainly, Leibniz' notation is not wrong and it may be used unscrupulously as long as one is always aware of the fact that $+s - \sigma$ is just another way of denoting a pair of numbers. But it must be admitted that the notation did cause some misunderstandings, concealing the formal structure of Leibniz' ideas[68].

One of Leibniz' few examples is the following:

sapiens: +20 –21 = +s - σ

pius: +10 - 3 = +p - π

Neither in this example nor in any other did Leibniz give any clue how to calculate the characteristic numbers[69]. He took them for granted and assumed that they possess the following property[70]: Only those "apt" (lt. *apti*) pairs are allowed, where s and σ are relative prime, i.e. where these numbers do not possess a proper common divisor. This definition gives us the clue that these „apt" number pairs may be looked upon as rational numbers:

Each *apt* pair of numbers corresponds to exactly one rational number if one associates s/σ with (s,σ). But the reverse of this proposition is also true: Given a rational number as a quotient

[65] We mention this fact because later on we are going to argue with respect to B, accordingly.

[66] Leibniz' remarks concerning negative numbers appear at the end of N. 59, the last manuscript in which models of type A are discussed.

[67] "Si qua offeratur propositio, tunc pro quolibet ejus Termino, subjecto scilicet vel praedicato, scribantur numeri duo, unus affectus *Nota*, +, seu *plus*, alter *Nota*, -, seu *minus*." N. 63, p. 243.

[68] Lukasiewicz is one of the few scholars who employed Leibniz' characteristic numbers for his own research into the depths of Aristotelian logic. He had a totally correct understanding of the significance of the „Notae" + and –, which may be inferred from the fact that he ignored them completely. This fact motivated Marshall in 1977 to comment Lukasiewicz' work as follows: „The interpretation that Lukasiewicz gives is not Leibniz' but a slightly modified variant. He drops the requirement that one of the numbers assigned by a term is negative." This remark in Marshall, 1977, Footnote on p. 239, shows that the author of those lines did in fact misapprehend Leibniz' idea.

[69] We will return to this point in our final Section 5.

[70] ‚Cavendum tantum ut duo numeri ejusdem Termini nullum habeant communem divisorum, nam si verbi gratia numeri pro *sapiente* essent +6-9, qui ambo dividi possunt per 3, nullo modo essent apti.' N. 62, p. 237, 6.

r/ρ, one cancels common factors of the denominator and the numerator (which does not alter the rational number), until in the final representation s/σ there are no common factors left. Then (s,σ) is an *apt* pair of numbers, being independent of the initial representation of the given rational number[71].

While it is *possible* to interpret the characteristic numbers of Model B in this manner as positive rational numbers, there is, of course, no absolute necessity to do so. It will depend on the special task at hand, whether one will really use this interpretation. If one had to choose between one of the following notations:

+s- σ,

(s,σ), and

s/σ

One would possibly use the first one for historical purposes, the second one in the context of philosophical logic and the last one in connection with theoretical investigations regarding the system of characteristic numbers. In this paper we will use all three notions in the appropriate context.

It is astonishing that Leibniz did not, to my knowledge, give an interpretation of his pairs of numbers as positive rational numbers. He came, however, very close to it: In N. 58, before constructing the number pairs, he experimented with quotients of terms, where the numerator denotes terms which are not contained in a another term:

> Given any fraction w/s, it can be said that w/s is the negation of any species of this s or a number divisible by s or of zs, or that it is the same as 'no s'. [72]

This sounds like a preparation for the introduction of a calculus where concepts are represented by rational numbers. However, Leibniz did not accomplish this last step. He resumed the basic idea of "term division" at different places of his work, but it is difficult to understand why he did not relate his important achievement, his system of pairs of characteristic numbers of Model B, to the operation of division and to rational numbers[73].

We now proceed to discuss Leibniz' interpretation of the four „Aristotelian relations" which he defined in his basic domain, the set of the *apt* pairs of natural numbers:

U.A. $\boldsymbol{a}((s,\sigma), (p,\pi))$ if and only if: p divides s and π divides σ.

P.A. $\boldsymbol{i}((s,\sigma), (p,\pi))$ if and only if: s and π as well as σ and p are relatively prime.

It is not necessary to write down the definitions for the negative propositions U.N. (***e*** – relation) and P.N. (***o*** – relation) because they are just negations of P.A (***i*** – relation) and U.A. (***a*** – relation) as required in Criterion 3. But we can say much more on the connection between the four relations: ***a*** and ***i*** are connected via Criterion 2 (ecthesis)[74]! In other words: If one defines the ***a*** – relation like Leibniz did, then the definition of ***i*** follows automatically. Thus the defini-

[71] Thus one obtains, f.i., always exactly the same apt number pair +2- 5, independently whether one starts from 2/5, 4/10, 16/40 or any other representations of the rational number 0.4. Another possibility of expressing this fact is: To each rational point of the positive semi- axes of the real number line there corresponds a unique apt pair of numbers, i.e. exactly one of the characteristic numbers of Model B.

[72] „Data quacumque fractione w/s dici potest w/s esse negationem cujuscumque speciei ipsius s sive numeri per s divisibilis sive ipsius zs seu idem esse quod *nullum* s." We shall soon see that this can be expressed by the fact that +w- s stands in relation ***e*** to +zs- 1.

[73] In a paper written much later in the context of his work on the algebra of logic he writes: "Thus I uncovered this secret on which, some years ago, I brooded in vane." (A, N. 165; FS, p. 285; C, p.386: "Ita arcanum illud detexi, cui ante aliquot annos frustra incubueram".) Then he defines his *propositiones* as follows: UA: A=AB; UN: A=A/B; PA: A≠A/B; PN: A≠AB. But he stops at this point and does not try to connect these formulas with his characteristic numbers.

[74] This cannot be realised by just looking at the definitions! We prove this fact in the Appendix of this paper.

tion of ***a***, together with the Criteria 2 and 3 fixes the whole model uniquely. For sake of lucidity we are going to summarize these considerations in the following theorem, the proof of which is banished to the Appendix.

Theorem 1. Leibniz Model B of Aristotelian logic, based on *apt* pairs of natural numbers, fulfills all model criteria: Criterion 1 (transitivity), 2 (ecthesis), and 3 (contradiction). It is the unique model built on the basic domain which can be constructed using Leibniz' definition of the *U.A. propositio* ***a***.

This implies that any other model must differ from Leibniz' Model B *either* by the basic domain *or* by the definition of the ***a*** – relation.

Now we are going to clarify the relation of Model B to the simpler Model A2: It is possible to embed A2 into B or, as one could reformulate this assertion, Model B is an extension of A2. Concerning the basic domain, this is quite clear, as the set of natural numbers (the domain of A2) is a subset of the set of rational numbers. The embedding is the usual one: just assign, to each natural number, the rational number s/1. In terms of number pairs: assign to s the *apt* pair (s,1) or, in Leibniz' notation, the characteristic number +s–1[75]. We have, however, still to prove that by this assignment all the Aristotelian relations defined on B turn out to be extensions of the corresponding relations on A2. This is quite clear for the central ***a*** – relation: If two pairs of numbers, (s,1) and (p,1) are related by ***a***, then, by definition, p divides s. This is in turn equivalent to the fact that s and p are related by ***a*** also in Model A2[76]. Now, according to the results on A2 in Section 3 and by Theorem 1, the other three relations are defined by ***a*** and Criteria 2, 3 in *both* models correspondingly. This proves the following[77]

Theorem 2. Model B is the unique extension of Model A2 into the set of rational numbers provided that ***a*** is defined as Leibniz suggested in his *U.A propositio.*

The „big" Model B has inherited all the positive properties of the smaller Model A2 (Criteria 1, 2, and 3), and - in contrast to the smaller model – it is also comprehensive enough[78]. Let us mention that in B all valid syllogisms of classical Aristotelian logic are also true[79]. In addition, syllogistic „figures" which are *not* valid in classical theory do not hold true in model B. This is an important result of Lukasiewicz[80].

In view of all this positive result concerning Leibniz' characteristic numbers, it is not obvious why Leibniz abandoned his work on his numbers so abruptly after having written his fragment N. 64. Anyhow, it is not true that he stopped working on this matter because he had detected an error in his considerations[81].

However, it may well be that Leibniz was dissatisfied with his construction regarding the topic of term negation. There is no doubt that he struggled with this problem in his last manuscript on the subject, but we may only speculate on whether this led to his capitulation, in view of the difficulties he encountered. In the last section we will go into the details of this

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[75] Due to the equation s=s/1, our way of denoting characteristic numbers by quotients of natural numbers is of special advantage, as the imbedding of A2 into B becomes obvious.

[76] We refrain from using different signs for the *U.A. propositio* in A2 and B, respectively.

[77] There are easy *direct* proofs for the relations ***i, e, o***, too. Using the definition of Leibniz' *P.A. propositio*, ***i***((s,1),(p,1)) does not restrict s or p in any way (as 1 does not possess proper divisors). But this is just the (problematic) property of the ***i*** – relation in Model A2: ***i***(s,p) is true for *all* s and p. Thus, from ***i***((s,1),(p,1)) in B it follows that ***i***(s,p) in A2. The corrresponding result for ***e*** and ***o*** follow by negation.

[78] In opposition to A2, there is an infinity of number (s,σ) and (p,π) in B which stand in the ***e*** – relation to each other: this is true for all pairs(s,σ) and (p,π) for which s and p or σ and π are not relatively prime, f.i., ***e***((6,35),(11,14)) holds because 6 and 14 posses the common divisor 2.

[79] We will not go into the details of this fact because, in this paper, we want to concentrate on the structure of Leibniz' model and not on the theory of syllogisms.

[80] One could also prove this result by providing a counterexample for each non valid syllogism in the way Thiel, 1980, did it for the "AOO form of the third figure".

[81] Cf. Henrich, 2002, who discusses the history of this wrong view on Leibniz' motives.

question, and we will discuss another problem related to the chance of success of his whole ambitious program.

5. Two Open questions

Since its publication at the beginning of the past century, Leibniz' arithmetic calculus has been the subject of a few philosophical-historical investigations. However, with the exception of Lukasiewicz' detailed study, it has been never used as an *instrument* of logical research. This is surprising, because Leibniz' arithmetic method is, after all, the direct predecessor of Gödel's arithmetization of first order predicate logic. With the following notes we attempt to stimulate the utilization of characteristic numbers as a tool for further research on in formal logic.

We will describe two problem areas in which open questions arise. Both questions have already been addressed in relevant literature; our goal is to formulate these problems in a precisely defined formal way so that their answer can be tackled independently of historical investigations.

The first question is directly linked to the conclusions of the preceding section. It concerns the problem of modelling negative terms in the context of model B. Leibniz had already experimented with term negation in connection with Model A2, and in Section 3 we saw why he could not succeed within this framework.

Within the context of the extended model B, Leibniz tried to model term negation as follows: Let

$$(s,\sigma)$$

or, in Leibniz notation, $+s-\sigma$, denote the characteristic number pair of a certain term (f.i., "man"). Then it is tempting to let

$$(\sigma,s)$$

(or, $+\sigma-s$) denote the „negative" of the corresponding term, "non-man". Unfortunately, this simple idea cannot be successful, as one realises by means of the following consideration[82]. For sake of formal purposes, let us define a "negation operator" N by

$$N(s,\sigma) := (\sigma,s).$$

We assume that, for two characteristic numbers $S=(s,\sigma)$ and $P=(p,\pi)$ the relation $\boldsymbol{a}(S,P)$ holds true. Assuming the validity of the classical Law of Contraposition, this is equivalent to $\boldsymbol{a}(N(P),N(S))$. Now, $\boldsymbol{a}(S,P)$ is, by definition, equivalent to

$$p \mid s \text{ and } \pi \mid \sigma,$$

whereas $\boldsymbol{a}(N(P),N(S))$ is equivalent to

$$\sigma \mid \pi \text{ und } s \mid p.$$

Of course, these divisibility conditions are not equivalent! Therefore, as even the fundamental Law of Contraposition does not hold, this way of modelling term negation fails.

Leibniz seems to have realised this problem[83]. However, at the end of his investigations on characteristic numbers he experimented with the negation of terms again. The corresponding lines in his last manuscript give the impression that he did not have a clear notion of how to proceed with this subject. He writes[84]:

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[82] Cf. Kauppi, 1960, who gives a slightly more complicated argument.
[83] N. 63, p. 249, 9: "De conversione per contrapositione hic non loquor. Ea enim novum terminum assumit." Here Leibniz refers to „negative terms", which he tries to avoid at this place.
[84] N. 64, p. 253.

Nullus homo est lapis

seu Omnis homo est non lapis.

Sit +h- c 1 - cd

debet h dividi per1, et c dividi per cd.

We interprete this lines as follows: With *homo,* the characteristic number pair *+h –c is* associated, and with *non lapis* the characteristic number *+1- cd.* Employing term negation in the way explained above, to *lapis* there belongs the characteristic number *+cd – 1.* Now one sees that Leibniz constructed this example in such a manner that „propositio universalis negativa" (*Nullus homo est lapis*) holds. For c is, of course, a proper divisor of cd[85]. However, *„Nullus homo est lapis"* is logically equivalent to *„Omnis homo est non- lapis"* which would, by definition of the *U.A. propositio,* imply that *cd* divides *c* – but this is certainly impossible[86]! The last line of the manuscript (*„debet .... c dividi per cd*") points very clearly to this contradiction, and Leibniz did not make any further attempt to attack this problem.

Let us stress again: All this is not a sign of a faulty system but merely indicates a certain *limit* of Model B in which the obvious way of coping with term negation does not work. Since, for Leibniz, term negation has always been an important subject, it is not inconceivable that he abolished the whole project of characteristic numbers because he could not successfully include this topic of negative terms into his theory.

Problem 1. Is it possible to *modify* Leibniz' system of characteristic numbers (Model B) in s such a way that *term negation* can be included[87]?

The second question we want to raise at the end of this paper concerns an important practical aspect of Leibniz' idea. At different passages of his manuscripts, Leibniz quotes very small examples, assigning numbers or pairs of numbers to terms. However, he never raises the question how to perform this assignment in general. Let us illustrate the corresponding problem with a simple example.

We consider a small number of propositions with four term constants *w, x, y,* and *z*: *A*(*x,w*), *A*(*y,x*), *O*(*z,x*), and *I*(*z,w*). The question is, how to find characteristic number pairs (or, rational numbers) *m, n, p,* and *q* corresponding to the four terms, so that the relations ***a***(*n,m*), ***a***(*p,n*), ***o*** (*q,n*), and i(*q,m*) hold? – This example – though larger than all the others which Leibniz and his commentators have discussed up to now – is nevertheless so small that one can find a solution just by trial and error. But up to now there is no *general* method for the task of assigning characteristic numbers to terms. Therefore we pose

Problem 2. Is it possible to construct an algorithm which, for any non contradictory set of propositions, computes a set of corresponding characteristic numbers?

We do not suggest that Leibniz abandoned his project of characteristic numbers because he did not know how to compute these numbers[88], but we maintain that this is a central problem. A positive answer to the question would have been a pre-requisite for the success of Leibniz' whole project.

[85] Which implies that the ***i*** – relation does not hold and thus the ***e*** – relation is valid.
[86] Parts of the same fragment are confused. Leibniz seems to struggle hard; he even experiments with expressions of the form - σ+s as well as - σ - s, which do not comply with his formalism of characteristic numbers. Here his own notation seems to lead him into formally problematic direction.
[87] It seems impossible to achieve this goal by just inventing a new kind of "negation of number pairs" which is different from the operator N defined above.
[88] There has been a controversial discussion on this subject, cf. Kauppi 1960, Henrich 2002.

5. Final remarks

The arithmetic calculus belongs to Leibniz' most important as well as most misunderstood achievements in the framework of his formal logic. With this paper, we are hoping to stimulate further work on this subject. While it is of course indispensable that philologically oriented philosophers deal with Leibniz' manuscripts, it would be a pity if the ideas of this ingenious formal logician did not also find entrance into the standard literature of formal logic[89]. Even if it is clear today that Leibniz grandiose idea of a *calculus universalis* will never be realisable, the results of Lukasiewicz nevertheless show clearly that the characteristic numbers represent a powerful instrument for the analysis of Aristotelic logic.

6. Appendix: Proof of Theorem 1

Theorem 1. Leibniz Model B of Aristotelian logic, based on *apt* pairs of natural numbers, fulfills all model criteria: Criterion 1 (transitivity), 2 (ecthesis), and 3 (contradiction). It is the unique model on the basic domain which can be constructed using Leibniz' definition of the U.A.- *propositio **a***.

Proof. We have only to show that Criteria 1, 2, and 3 are fulfilled in Model B. As Criteria 2 and 3 define the relations ***i, o*** und ***e*** *uniquely* by means of relation ***a***, everything will be proved. Now, Criterion 1 (transitivity) and Criterion 3 (contradiction) are fulfilled for *all* models constructed by Leibniz, it suffices to deal with Criterion 2 (ecthesis). Here we have to show that, defining ***i*** by ecthesis, we will arrive exactly at Leibniz' definition of the „*P.A. propositio*", and vice versa.

Thus, let us assume that, for any two pairs of characteristic numbers (s,σ) and (p,π), ***i*** is connected to the fundamental relation ***a*** by ecthesis:

i((s,σ), (p,π)) if and only if there exists (z,ζ) so that

a((z,ζ), (s,σ) *) and* ***a***((z,ζ), (p,π)).

Here we take for granted that all number pairs are *apt* i.e., they consist of components which do not have a proper common divisor.

From ***a***((z,ζ), (s,σ)) we deduce the existence of two numbers m, μ such that

$z = m\, s$ and $\zeta = \mu\, \sigma$, (*)

and from ***a***((z,ζ), (p,π)) there follows the existence of two numbers n, ν with

$z = n\, p$ and $\zeta = \nu\, \pi$. (**)

We have to prove, corresponding to the condition in Leibniz' definition of the *P.A. propositio,* that s and π as well as σ and p are relatively prime, respectively. Let us assume to the contrary that this is not true for s and π[90]! Then there is a number $q \geq 2$ so that

[89] Even if the objectives of Gödel's investigations on the arithmetization of first order predicate logic were completely different, there is a strong relation of Gödel's and Leibniz' basic method. Thus one should not speak about the concept of *Gödelization* of predicate logic without mentioning the method of *Leibnitisation* of Aristotelian logic. Incidentally, Gödel admired Leibniz for his logical work which he knew quite well.

[90] For reasons of symmetry we must only consider this case.

$s = e\,q$ and $\pi = d\,q$ [91].

Thus, because of the first part of (*),

$$z = m\,e\,q,$$

and because of the second equation of (**),

$$\zeta = \nu\,d\,q.$$

Therefore, in contradiction to the assumption, z und ζ are not relatively prime, i.e. the number pair (z,ζ) is not *apt*. This finishes our proof by contradiction.

Concerning the reverse direction, we start from the definition of the *P.A. propositio* (***i*** – relation) and deduce the ecthesis condition appearing in Criterion 2. Let ***i***((s,σ), (p,π)) hold true if and only if s and π as well as σ und p possess no proper common divisors, repectively (Leibniz' definition the P.A. – *propositio*). We are going to show that this fact implies that, for (s,σ) and (p,π), the condition of ecthesis (Criterion 2) holds. We shall accomplish this by explicitly constructing a pair (z,ζ) required in this Criterion as follows:

$z = s\,p$ and $\zeta = \sigma\,\pi$.

By construction, (z, ζ) fulfills ***a***((z,ζ), (s,σ)) and ***a***((z,ζ), (p,π)). Thus it remains to be shown that (z, ζ) is an *apt* pair of numbers; i.e. that z and ζ have no proper common divisor. Let us assume to the contrary that d is a prime number dividing z as well as ζ. Then, by definition of z, d is a divisor of s or p. Let us assume, without restriction of generality, that d divides s. Now, by definition, d divides ζ too, i.e. it divides σ or π. But d *cannot* divide σ because otherwise s and σ have a proper common divisor – which is impossible as (s, σ) is an *apt* pair of numbers. Therefore, d divides π – in contradiction to the assumption (definition of the *P.A. propositio*). This contradiction proves the assertion.

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[91] The other possibility, $\pi = q\,s$, may be neglected for reasons of symmetry.

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