ON NEGATION IN LEIBNIZ' SYSTEM OF CHARACTERISTIC NUMBERS

KLAUS GLASHOFF

1. Introduction

In the spring of 1679, Leibniz invented a famous interpretation of Aristotelian logic by means of his *characteristic numbers* (see Glashoff [Gla02] and the references given there). Leibniz was able to show that, within his model, all classical laws of "positive" Aristotelian logic (a term logic without negation) hold, if one uses a certain proper arithmetical interpretation of the Aristotelian quantors A, E, I, and O. While this construction of characteristic numbers is a highly esteemed result today[1], Leibniz himself was apparently not content with his achievement. His last notes on this subject show how hard he struggled with different unsuccessful attempts of allowing also for term negation within his formalism. Later on he never resumed his work on characteristic numbers.

The question of how to extend the system of characteristic numbers in order to include term negation has not yet been solved.[2]

After this introduction, we will describe, in Section 2, Leibniz' original system $\mathcal{C}_+$ of characteristic numbers in modern formal setting.

We will then, in Section 3, show *why* Leibniz' attempts were bound to fail: It is *impossible* to define, on the set $\mathcal{C}_+$, a negation operator

$$\nu : \mathcal{C}_+ \to \mathcal{C}_+$$

which complies with the Aristotelian *Law of Contraposition*.

After having discussed this negative result, we will show how to complete Leibniz' construction of characteristic numbers by extending it from *positive* to *full* Aristotelian logic in a formal correct way. Because of the above mentioned negative result, this construction[3] requires an appropriate embedding of $\mathcal{C}_+$ into a larger set $\mathcal{C}$.

2. Leibniz' characteristic numbers: Positive logic

In this section, we present Leibniz' system of characteristic numbers for Aristotle's positive logic of the *Prior Analytics*. Let us first remark that characteristic numbers belong to the sphere of *semantics* of Aristotelian logic: Leibniz' number system is a concrete domain of (pairs of) numbers which will be equipped with

[1] The system of characteristic numbers is a "non-extensional model" of Aristotelian logic in the sense that it does not make use of the concept of individuals as basic entities.

[2] Sotirov's claim of having solved this problem is misleading, because he deals only with the special case of *finitely* many basic building blocks (wheras, in Leibniz' arithmetisation, these building blocks are taken out of the *infinite* set of prime numbers) ([Sot99]).

[3] Our method utilizes a formal construction used by Sheperdson [She56], who used it in his completeness proof for Lukasiewicz' formal system of Aristotelian logic.

four concrete relations. These relations, a, e, i, and o derive directly from the divisibility relation within the set of positive integers. In an interpretation of a set of Aristotelian categorical propositions of type Ax_1y_1, Ex_2y_2, Ix_3y_3 and Ox_4y_4, *terms* x_i, y_k will be mapped at corresponding number pairs, and the Aristotelian quantors A, E, I, and O will be mapped at the corrisponding relations a, e, i, and o.

Leibniz' idea was to represent *basic concepts* by prime numbers and compound concepts by products of prime numbers. This simple first idea[4] served as a foundation for his more sophisticated construction of characteristic numbers as *pairs* of integers:

Definition 2.1. Let $\mathbb{N}^2$ denote the set of pairs of positive integers. The set of characteristic numbers, $\mathcal{C}_+$, is defined by

$$\mathcal{C}_+ = \{(m, \mu) \in \mathbb{N}^2 \backslash (1,1), \text{ where } m \text{ and } \mu \text{ are relatively prime}^5\}.$$

Comment 2.1. It makes sense, for some purposes, to reduce the set of characteristic numbers by requiring m, μ to belong to the subset $\mathbb{N}_{sqf}$ of *squarefree*[6] integers.

Comment 2.2. It is suggesting to identify characteristic numbers with a certain subset of the positive rationals by using the standard imbedding

$$(m, \mu) \to m/\mu.$$

We will not make use of this embedding at this moment, but we will come back to this interesting possibility.[7]

After having defined the basic set $\mathcal{C}_+$ of characteristic numbers, we will now define the four relations a, e, i, and o on $\mathcal{C}_+$ which - on the semantic side - serve as interpretations of the classical Aristotelian quantors A, E, I, and O (which live in the world of Aristotelian syntax).

Definition 2.2. For $(s, \sigma), (p, \pi) \in \mathcal{C}_+$, let

$$a((s, \sigma), (p, \pi)) := (p \mid s) \wedge (\pi \mid \sigma)^8$$

Definition 2.3. For $(s, \sigma), (p, \pi) \in \mathcal{C}_+$, let

$$e((s, sigma), (p, \pi)) := gcd(s, \pi) \times gcd(p, \sigma) > 1^9$$

The remaining two relations, i and o are negations of e and i, respectively.

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[4] This method was not complex enough for building a "rich" model of Aristotelian logic, see [Gla02]

[5] "The integers a and b are said to be coprime or relatively prime if and only if they have no common factor other than 1 and -1, or equivalently, if their greatest common divisor is 1. ... The number 1 is coprime to every integer." (http://en.wikipedia.org/wiki/Coprime).

[6] "The integer n is square-free if and only if in the prime factorization of n, no prime number occurs more than once. Another way of stating the same is that for every prime divisor p of n, the prime p does not divide n / p. Yet another formulation: n is square-free if and only if in every factorization n=ab, the factors a and b are coprime." (http://en.wikipedia.org/wiki/Squarefree)

[7] See [Gla02]

[8] Which means: p divides s and π divides σ.

[9] Which signifies: "s and π have a common divisor which is greater than 1, *or* p and σ have a common divisor which is greater than 1". The greatest common divisor (gcd), sometimes known as the greatest common factor (gcf) or highest common factor (hcf) of two integers which are not both zero is the largest integer that divides both numbers.

Definition 2.4. For $(s,\sigma),(p,\pi) \in \mathcal{C}_+$, let

$$i((s,\sigma),(p,\pi)) := gcd(s,\pi) \times gcd(p,\sigma) = 1.$$

This means " There is no proper common divisor[10] of s and π, *and* there is no proper common divisor of p and σ."

Definition 2.5. For $(s,\sigma),(p,\pi) \in \mathcal{C}_+$, let

$$o((s,\sigma),(p,\pi)) := \neg(p \mid s) \vee \neg(\pi \mid \sigma).$$

Let us introduce a notation which helps to make clear the abstract background of all the following constructions.

The relation a defines a *partial order structure* on $\mathcal{C}_+$. Thus, we will define

$$(s,\sigma) \preceq (p,\pi) := a((s,\sigma),(p,\pi)) \tag{2.1}$$

Definition 2.6. (meet relation) Let $(\mathcal{O},\preceq)$ be a partial order. The *meet relation* $\downarrow$ belonging to$(\mathcal{O},\preceq)$ is given by

$$x \downarrow y \longleftrightarrow \textit{iff} \text{ the set } x,y \text{ has a lower bound}^{11}\ .$$

In [Gla02] we have proven the following Lemma which says that, in Leibniz' system, the relations a and i of Definitions (2.2) and (2.4) are connected by *ecthesis*:

Lemma 2.7. $i((s,\sigma),(p,\pi)) = (s,\sigma) \downarrow (p,\pi)$.

Thus, the definition of the i- relation on $\mathcal{C}_+$ has the properties of *ecthesis* with respect to the partial order $\preceq$: $i((s,\sigma),(p,\pi))$ holds *iff* $(s,\sigma),(p,\pi)$ have a lower bound in $\mathcal{C}_+$ with respect to the partial order $\preceq$ defined in Definition (2.1).

Thus, Leibniz' definition of the Aristotelian relations a, i, e, and o is a special case of the definition of such relations in an arbitrary partially ordered structure $(\mathcal{O},\preceq)$:

Definition 2.8. (Aristotelian relations on a partially ordered set $(\mathcal{O},\preceq)$)

(1) $a(x,y) := x \preceq y$;
(2) $i(x,y) := x \downarrow y$;
(3) $e(x,y) := not(x \downarrow y$;
(4) $o(x,y) := not(x \preceq y)$

Leibniz' set $\mathcal{C}_+$ of characteristic numbers is an ideal domain for semantics of positive Aristotelian logic. Łukasiewicz [Lu57] made heavy use of it in connection with a completeness theorem for his syntactical system, which is a mixture of term logic and propositional logic. Maldonado (see [Mal98]) proved a completeness theorem, using Corcoran's transcription of Aristotelian logic and Leibniz' characteristic numbers.

This leads to a general *non-extensional* model[12] theory for Aristotelian logic, where the class of model structures is simply the class of partially ordered sets (see [Mar97], who virtually proved a completeness theorem for this setting, albeit in slightly deviating axiomatic surrounding).

[10]A divisor d is called proper, if $\mid d \mid> 1$.

[11]viz i.e. there is an element $z \in \mathcal{O}$ such that $z \preceq x$ and $z \preceq y$

[12]Let us call a model of Aristotelian logic non-extensional, if the construction of an interpretation of a set of canonical propositions does not require the precedent construction of *individuals*.

3. The difficulties of term negation

Let us summarize what has been achieved up to now: Utilizing Leibniz' ideas of modelling Aristotelian syllogistic by means of the set $\mathcal{C}_+$, we have created a concrete domain of (pairs of) natural numbers in which all laws of Aristotle's positive syllogistic[13] hold. This set $\mathcal{C}_+$ of numbers is a partial ordered set by means of the relation $\preceq$ defined in Definition (2.1).

Leibniz' idea was the following: He proposed to search for an infinite number of basic philosophical and scientific concepts which he would represent by the sequence of prime numbers $2, 3, 5, 7, 11, ...$ He then defined *compound concepts* by two products of prime numbers. The idea behind it is that, if (p, π) is a compound concept, then the set of prime factors of p contains the basic concepts which are contained in (p, π), and that the set of prime factors of π contains the basic concepts which are definitely *not* contained in (p, π).

Thus, (p, π) is an arithmetization of a conjunction of two propositions:

(1) The compound concept denoted (p, π) contains the basic concepts belonging to the prime factors of p.
(2) The compound concept denoted (p, π) *does not contain* the basic concepts belonging to the prime factors of π.

Leibniz then tried to negate this conjunction of propositions: He defined a negation function $\nu_L : \mathcal{C}_+ \to \mathcal{C}_+$ by

$$\nu_L(p, \pi) := (\pi, p), \tag{3.1}$$

He soon realized that this definition did not behave well in terms of the standard properties of term negation (for example, contraposition does not hold for this definition).

The reason is the following: The negation of the conjunction defined above is not again a conjunction of the same type!

One could try to alter Definition (3.1), i.e., of the negation function. However, because of the following property of Leibniz' system $\mathcal{C}_+$, this route cannot be successful.

Theorem 3.1. *There does not exist a bijection*

$$\nu : \mathcal{C}_+ \to \mathcal{C}_+$$

for which the law of contraposition,

$$a((s, \sigma), (p, \pi)) = a(\nu(p, \pi), \nu(s, \sigma)), \tag{3.2}$$

holds for any pair of characteristic numbers (s, σ) and (p, π).

Proof. Let $\preceq$ denote (see (Definition 2.1)) the a – relation on $\mathcal{C}_+$, and let $\prec$ denote the *strict part* of $\preceq$. $\succeq$ and $\succ$ are the converses of $\preceq$ *and* $\prec$, respectively. Let $(p_1, \pi_1), (p_2, \pi_2), ...$ denote an infinite number of *different* characteristic numbers, which are nonincreasing[14] with respect to the partial order $\preceq$.

$$(p_1, \pi_1) \succeq (p_2, \pi_2) \succeq (p_3, \pi_3) \succeq ... \tag{3.3}$$

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[13] Aristotles syllogistics of the Prior Analytics without term negation

[14] Such sequences exist, e.g.

$$(2, 1) \succeq (2 \cdot 3, 1) \succeq (2 \cdot 3 \cdot 5, 1), ...$$

As a matter of fact,

(3.4) $$(p_1, \pi_1) \succ (p_2, \pi_2) \succ (p_3, \pi_3) \succ ...$$

holds, because all number pairs were assumed to be different from each other. This sequence gives rise to the corresponding sequence

$$\nu(p_1, \pi_1), \nu(p_2, \pi_2), \nu(p_3, \pi_3)...$$

By (3.2) (contraposition) and (3.3) we obtain

$$\nu(p_1, \pi_1) \preceq \nu(p_2, \pi_2) \preceq \nu(p_3, \pi_) \preceq ...$$

ν is injective, therefore

$$\nu(p_i, \pi_i) \prec \nu(p_k, \pi_k), i < k,$$

which implies

(3.5) $$\nu(p_1, \pi_1) \prec \nu(p_2, \pi_2) \prec \nu(p_3, \pi_3) \prec ...$$

Let

(3.6) $$(\bar{p}_i, \bar{\pi}_) := \nu(p_i, \pi_i).$$

Then

$$(\bar{p}_i, \bar{\pi}_i) \prec (\bar{p}_k, \bar{\pi}_k)$$

implies, by Definition (2.2),

$$\bar{p}_k \mid \bar{p}_i, \textit{ and } \bar{\pi}_k \mid \bar{\pi}_i,$$

This gives rise to two nonincreasing sequences of natural numbers,

(3.7) $$\bar{p}_1 \geqslant \bar{p}_2 \geqslant \bar{p}_3 \geqslant ...$$

and

(3.8) $$\bar{\pi}_1 \geqslant \bar{\pi}_2 \geqslant \bar{\pi}_3 \geqslant ...$$

Because of (3.5), $\bar{p}_i > \bar{p}_{i+1}$ or $\bar{\pi}_i > \bar{\pi}_{i+1}, i = 1, 2, ...$ (or both). Thus, the sequences (3.7) and (3.8) will stop (become constant) in at most $(m + \mu) - 1$ steps[15]. This is in contradiction the assumption that all of the the elements of the sequence (3.3) are different.

□

This negative result implies that there are exactly two different possibilities of introducing a proper term negation function for Leibniz' characteristic numbers:

(1) *Considering only a finite set of characteristic numbers.* It is then possible to define negation functions with all required properties, including contraposition. This is what Sotirov proposed in [Sot99] and a whole series of other papers. Evidently, this is *not* what Leibniz intended, who *always* considered characteristic numbers built up from the set of *all* (infinitely many) prime numbers.
(2) *Embedding the set of characteristic numbers* together with the relations a, e, i, and o into a larger set of objects, within which term negation raises no problems. This is the idea which we will accomplish in the final section of this paper.

[15]Mathematically spoken: The componentwise order $\leqslant_2$ on $\mathbb{N}^{\not\vDash}$ is *well founded.* This implies that there is, with respect to $\leqslant_2$, no decreasing infinite sequence of number pairs.

4. How to define term negation of characteristic numbers

As has been said above, Leibniz himself did not succeed with term negation of characteristic numbers. He defined a negation Function ν_L by

$$\nu_L(p,\pi) := (\pi, p), \tag{4.1}$$

and, constructing some examples, he realized that this definition did not work.

The limits of improving his results are, in view of Theorem (3.1), clear. If we want to retain Leibniz' original ideas of

- infinitely many characteristic numbers,
- defining the a-relation by means of simple divisibility as in (2.2),

then we have to *enlarge* the system.

Definition 4.1. ($\langle p,\pi\rangle$, $\mathcal{C}_-, \mathcal{C}$)

(1) To each characteristic number (p,π) of $\mathcal{C}_+$, define a new symbol $\langle p,\pi\rangle$.
(2) Define $\mathcal{C}_- = \{\langle p,\pi\rangle/(p,\pi) \in \mathcal{C}_+\}$.
(3) Let $\mathcal{C} := \mathcal{C}_+ \cup \mathcal{C}_-$

Now it remains to define a term negation function

$$non : \mathcal{C} \to \mathcal{C}$$

and also to extend the four *Aristotelian relations* a, e, i, and o from $\mathcal{C}_+$ to the whole of $\mathcal{C}$. It is very obvious how to define term negation:

Definition 4.2. (Term negation $non : \mathcal{C} \to \mathcal{C}$)

(1) $non(p,\pi) := \langle p,\pi\rangle$
(2) $non\langle p,\pi\rangle := (p,\pi)$

Because of Definition (4.1,(3)) the negation function *non* is completely defined.

Concerning the definition of the a-relation, there have to be considered *four different* cases, out of which the first two ones are, obvious, too:

Definition 4.3. ($a((s,\sigma),(p,\pi)), a(\langle s,\sigma\rangle,\langle p,\pi\rangle)$)

(1) On $\mathcal{C}_+ \times \mathcal{C}_+$: $a((s,\sigma),(p,\pi) := (p \mid s) \wedge (\pi \mid \sigma)$ (see Definition 2.2).
(2) On $\mathcal{C}_- \times \mathcal{C}_-$: $a(\langle s,\sigma\rangle,\langle p,\pi\rangle) := a((p,\pi),(s,\sigma))$.
(3) On $\mathcal{C}_+ \times \mathcal{C}_-$: $a((s,\sigma),\langle p,\pi\rangle) := e((p,\pi),(s,\sigma))$.
(4) On $\mathcal{C}_- \times \mathcal{C}_+$: $a(\langle s,\sigma\rangle,(p.\pi)) = FALSE$

This is exactly the imbedding given by [She56] who used it as a means for proving his theorem on the construction of set-models for Łukasiewicz – type transcriptions of Aristotelian logic.

Example 4.1. These are simple illustrations of the three first cases of the last definition.

(1) $a((10,21),(5,7))$ (because 5 divides 10, and 7 divides 3);
(2) $a(\langle 2,3\rangle,\langle 10,21\rangle)$ (because 2 divides 10, and 3 divides 21);
(3) $a((15,14),\langle 21,26\rangle)$ (because 14 and 21 have the common divisor 3).

Comment 4.1. There are other possible notational systems for $\mathcal{C}$, the original system characteristic numbers.

- Leibniz wrote $+p-\pi$ instead of our (p,π). The signs + and − serve as tags for marking the first and the second component of the pair (p,π); they have no arithmetical significance. Leibniz experimented with term-negation, trying to define $non(+p-\pi)$ by $+\pi-p$. We know now that this couldn't work (Theorem (3.1)). In one of his unpublished papers, he seems to experiment with "numbers" of type $-p+\pi$, which could perhaps be seen as an attempt of enlarging the set of characteristic numbers in order to be able to calculate with negated terms. Indeed: If one would define $\mathcal{C}_+$ by expressions of type $+p-\pi$ and $\mathcal{C}_-$ by expressions of type $-p+\pi$, then this would allow all the constructions we obtained with our notation (p,π) and $\langle p,\pi\rangle$. The only problem of this notational system is that the tags "+" and "-" suggest a meaning which they really don't posses in this context.
- The system $\mathcal{C}_+$ is isomorphic to the set of positive rational numbers which are different from 1. The isomorphism is given by

$$\phi : (p,\pi) \mapsto p/\pi \tag{4.2}$$

(see [Gla02]). How to denote, after having imbedded $\mathcal{C}_+$ in $\mathcal{C}$, the images of $\langle p,\pi\rangle$ under the extension of ϕ? One idea is the following: Define *antinumbers* by expressions of type

$$p\backslash\pi. \tag{4.3}$$

which are the images of p/π under the extension of ϕ. By means of this notation,

$$non(p/\pi) = p\backslash\pi, non(p\backslash\pi) = p/\pi. \tag{4.4}$$

For example, the following relations hold:

$$10/21 \preceq 5/3 \preceq 6\backslash 7 \preceq 12\backslash 77. \tag{4.5}$$

Theorem 4.4. *Let $\mathcal{C}$ be the following set of characteristic numbers*

$$\mathcal{C} := \{(p,\pi), \langle p,\pi\rangle / p,\pi \in \mathbb{N}, p \, and \, \pi \, coprime\}\},$$

equipped with the negation function non, and with the relations a, i, e, and o as defined in this Section. This system has the following properties:

(1) *$\mathcal{C}_+$ is a model for Corcoran's* [Cor73] *system of natural deduction*[16].
(2) *The Laws of Obversion and of Contraposition hold:*
  (a) $e((s,\sigma),(p,\pi)) = a((s,\sigma), non(p,\pi))$
  (b) $a((s,\sigma),(p,\pi)) = a(\nu(p,\pi), \nu(s,\sigma))$
(3) *For $x,y,z \in \mathcal{C}$, the following properties of* non *and a hold*[17]*:*
  (a) $a(x,x)$
  (b) $a(x,y)$ *and* $a(y,z)$ *imply* $a(x,z)$
  (c) $a(x,y)$ *implies* $a(nony, nonx)$
  (d) *Not* $a(x, notx)$

[16]With the difference that here *A(x,x)* is not a syntactically "forbidden proposition".

[17]Thus $\mathcal{C}$ is the model of a system discussed in [She56]. Shepardsen showed that the corresponding axiom system allows also set theoretic models.

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Vicolo Centrale, CH 6900 Lugano, Switzerland